Control Design for a Tactical Missile Autopilot

an Application of Robust Control

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Abstract

This study presents a holistic design procedure for an autopilot controller of a tactical SAM missile. First, the missile dynamics were modelled using the most realistic non-linear model available, and then a simpler linear model was constructed to be the basis of the control design.

An autopilot controller using the $H_{\infty}$ framework was designed, and its order reduced from 6th order into 2nd order. The resulting controller's performance was evaluated both in terms of theory and simulations.

Both the autopilot and the guidance system were included in the most realistic missile model, which was used in evaluating the performance of the resulting control system. This way, a holistic design for the autopilot was achieved, as well as the benefits of the design procedure were proven. Also the most crucial performance criteria were screened.
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1 Introduction

1.1 Tactical missile systems and control theory

Tactical missiles offer an interesting application for control systems. The missile systems can be divided into two functional subsystems, which can again be studied independently, the other half abstracted away. Problem division was essential also in this study, where the autopilot system was designed by first defining the interface between the two subsystems.

Aerodynamics of the missile result in highly complex mathematical models. This complexity can be reduced using simpler models and linearization, among other methods of model reduction. The necessity of these reductions pose on their part another challenge to the control system, requiring a strong level of robustness of the controller. The missile is also much affected by external noise, especially wind conditions.

Simulations become an important method in studying the missile control system's performance, since experiments in real conditions are costly and always include a military aspect. Due to the aerodynamic complexity and coordination transformations, even the formulation of the simulation model is a demanding task.

1.2 The purpose of this study and exclusions made

The purpose of the study was to model the dynamics of a tactical missile, design a control system using robust control design methods and finally implement this controller into a simulation model and study the controller performance in realistic tasks by simulations. Specific concern was laid on the effect of various model reductions to the performance of the control design and the resulting missile. The goal of this study was, accordingly, a rather holistic control design, validated with extensive simulations made using the most realistic model available.

The study was restricted in the vertical plane, so the aerodynamics were described only in two dimensions, and the rotational motion of the missile represented by only the pitch rate. This approach was chosen in order to maintain the problem complexity in a convenient level. The study could later be extended into three dimensions in a quite straightforward way.

The missile was assumed to be a surface-to-air (SAM) missile capable to velocities faster than the target, and no decision theoretical aspects of the pursuit situation was modeled. This was a conscious simplification, creating a sensible framework for studying the controller performance.

In addition, the missile sensor and actuator dynamics were either neglected or highly simplified, thereby concentrating only on the performance of the designed control system.
1.3 Overview of the missile system

Considering this study, the most important subsystems of the missile are the autopilot system and the command guidance system. The purpose of the latter is to generate guidance signals, most often in terms of desired normal acceleration or pitch rate, so that the flight path of the missile is optimal. Optimality in this case means most often minimum miss distance, minimum flight time or the two of these combined. The autopilot system gets the optimal command signal from the guidance system as reference, and tries to make the missile track it using the control surfaces.

A block diagram of the missile system is shown in figure 1. Other important subsystems include the observation system and the warhead of the missile, but these are not considered in this study. Instead, the important state variables are simply considered measurable, and target destruction is assumed within a specified distance from the missile.

![Missile System Block Diagram]

Figure 1: The missile system block diagram.

The inputs of the guidance system are the missile and target positions and velocities; the guidance system is discussed in more detail in section 4. The output of the guidance system is a signal representing the optimal normal acceleration (or pitch rate).

The autopilot system uses this reference as its input, and produces the command signal to the actuator surfaces. These surfaces control the roll, pitch and yaw rate, respectively. Because this study is restricted in the vertical plane, only the elevator surface controlling the pitch rate is taken into account. Thus, our autopilot becomes a SISO system.
2 Models Used in the Study

The study involves two different mathematical models of the missile dynamics, and yet one more model, which includes also the target motion and is used for the guidance system design.

2.1 The "Real Model"

2.1.1 General

The most realistic model was based on a six degree-of-freedom model of the missile aerodynamics, presented in [1, p. 33]. In addition to the missile's longitudinal motion, the rotational dynamics were modelled. This was done by taking the full model of the missile dynamics presented and dropping out longitudinal motion in the y axis direction. Rotational motion was described only as pitch rate, that is, the rotation in the xz-plane.

The lift and drag coefficients of the aerodynamic force resultant were modelled as nonlinear. The amount of fuel decreases as a function of the thrust force, so the mass of the missile does not remain constant in our model.

The airframe was, however, considered constant, because in the operational height of the simulated missions (0-500 m) the effect of change in the air density was quite minimal. The air density was chosen to be at the sea level according to the ARDC \(^1\) model atmosphere referred in [1, app. D].

2.1.2 The Equations of Motion

The missile equations of motions were first described in terms of the missile body axis. In the simulation model, these state variables are then transformed to the world reference axis to achieve comparability with the target motion.

The missile is affected by the aerodynamic force, which can be divided into lift and drag coefficients. In addition to these, it is affected by the gravity and the thrust force produced by the missile's rocket engine. The missile and the forces (in relation to the earth reference axis) are described in figure 2.

The flight path angle \(\gamma\) depicts the angle between the positive earth referenced \(x\) axis and the missile velocity relative to the airframe. Angle-of-attack \(\alpha\) is the angle between the longitudinal axis of the missile and the velocity, and the pitch angle \(\theta\) is the angle between the longitudinal axis and the positive body axis referenced \(x\) axis, respectively.

The lift force can be described as a product

\[ L = qSC_L(\alpha) , \]

\(^1\)Air Research and Development Command, of the U.S. Air Force
Figure 2: Forces affecting the missile with respect to the earth referenced axis.

where \( q \) is the dynamic pressure, defined as

\[
q = \frac{1}{2} \rho V^2 ,
\]

(2)

and \( S \) is the missile surface area. Lift coefficient \( C_L = C_L(\alpha) \) is a nonlinear function of the angle-of-attack.

Drag force is defined correspondingly as

\[
D = qSC_D(\alpha) ,
\]

(3)

where the drag coefficient can be expressed in terms of the zero drag coefficient and the lift coefficient [1, p. 54-61]:

\[
C_D(\alpha) = C_{D_0}(V) + KC_L^2(\alpha) .
\]

(4)

Now referring to figure 2 we can write the forces and moments acting on the missile body. The forces along the body reference axis are

\[
\begin{align*}
F_L &= T + L\sin(\alpha) - D\cos(\alpha) - G\sin(\theta) \\
F_N &= -D\sin(\alpha) - L\cos(\alpha) + G\cos(\theta)
\end{align*}
\]

(5)

The moments include the moments resulting from angle-of-attack and elevator surface angle (elevation angle), and also the change in the position of the missile's
The center of gravity moves because the amount of fuel, and this way the weight of the fuel tank, decreases as the thrust engine is used. The force causing this moment is the normal force resultant, and can accordingly be expressed as

$$F_M = D \sin(\alpha) + L \cos(\alpha) - G \cos(\theta).$$  \hspace{1cm} (6)

The torque arm here is the difference between the referenced center of gravity and the actual center of gravity that moves as a function of time:

$$c_{cg} = d_{cg}(t) - d_{cg}^{ref}.$$  \hspace{1cm} (7)

Here \(d\) denotes the distance from the tip of the missile, and is negative, \(d_{cg}^{ref}\) is the initial center of gravity. [1, p. 62-70]

The moment of inertia being denoted with \(I_y\), the total moment due to change in the center of gravity becomes

$$M_{cg} = \frac{1}{I_y}(d_{cg} - d_{cg}^{ref})(D \sin(\alpha) + L \cos(\alpha) - G \cos(\theta)).$$  \hspace{1cm} (8)

For determining the moment of inertia for rotation in the \(xz\)-plane, the missile can be considered as a uniform rod that has the axis of rotation at its center. The moment of inertia then becomes

$$I_y = \frac{1}{12}ml^2,$$  \hspace{1cm} (9)

where \(l\) is the length of the missile.

The moments due to angle-of-attack and elevation angle are more complicated to derive, and usually the nonlinear forces are determined using wind-tunnel tests [1, p. 63]. For this study these moments can be simplified using equations [2, p. 233-238]

$$\begin{cases} M_{\alpha} = c_{\alpha} \hat{C}_\alpha \cdot \alpha \\ M_{\delta} = -c_{\delta} \hat{C}_\delta \cdot \delta \end{cases}.$$  \hspace{1cm} (10)

where \(c\) is the torque arm, and \(\hat{C}\) is a coefficient representing the relation of the angle and the force resulting from that angle.

The missile is assumed to have been designed so that \(M_{\delta}\) has the opposite sign than \(M_{\alpha}\). This can be accomplished by placing the center of pressure between the center of gravity and the tail of the missile, and this condition is actually a prerequisite for achieving a stable missile. [1, p. 80]
2.1.3 The State-space Model

The state variables can be chosen as the velocity of the missile in direction of the longitudinal and lateral missile axis, the rotational (pitch) rate, and the corresponding angle. When simulating the missile flight, these state variables have to be transformed from body axis coordinates into earth reference axis. This simulation procedure is illustrated in figure 3.

![Figure 3: Flow chart of the simulation procedure.](image)

The pitch rate has to be coupled into the body axis state-space equations. Using this coupling and combining the prementioned equations for longitudinal and rotational motion, we obtain [1, p.28-33]

\[
\begin{align*}
\dot{u} & = -Qw + \frac{1}{m}(T + L\sin(\alpha) - D\cos(\alpha)) - g\sin(\theta) \\
\dot{\alpha} & = \frac{1}{mL}(D\cos(\alpha) + L\cos(\alpha) - mg\cos(\theta)) \\
\dot{Q} & = qS\left[Ma + M\delta + \frac{1}{T}(d_{cg} - \delta\epsilon_f)(D\sin(\alpha) + L\cos(\alpha) - mg\cos(\theta))\right] \\
\dot{\theta} & = \frac{1}{Q} \\
\end{align*}
\]

(11)

\(u\) representing the velocity in the (body axis) \(x\) direction, \(w\) the velocity in the \(z\) direction, \(Q\) the pitch rate and \(\theta\) the pitch angle, respectively.

Note that the inputs to this body axis model include the airspeed \(V_{air}\), the angle-of-attack \(\alpha\) and the controls (the throttle force \(T\) and the elevation angle \(\delta\)).

In addition to the body axis model, we need dynamics for the actuators and the missile mass reduction during the flight for the model to be complete. The actuators can be represented as first order dynamics:

\[
\begin{align*}
\dot{T} & = \frac{1}{T}\left(u_T - T\right) \\
\dot{\delta} & = \frac{1}{\delta}\left(u_\delta - \delta\right)
\end{align*}
\]

(12)
where $\tau_i$ is a constant describing the settling time, and $u_i$ the input signal.

The thrust force output is assumed to be linearly related to the mass of the fuel burned at each unit of time. The dynamics for the missile mass now follow

\[ \dot{m} = K_P T, \quad (13) \]

where $K_P$ is a constant describing the fuel energy content. The mass dynamics is restricted by inequality $m_{empty} < m(t) < m_{full}$.

Elevation angle is constrained by $|\delta(t)| \leq \delta_{max}$, and the throttle force by $T(t) < T_{max}$.

### 2.1.4 Numerical values

Model coefficients and their numerical values were combined from [1] and [3], and are presented in table 1. Mach number $M(v)$ in the lift and drag coefficients is defined as the relation of missile velocity and the speed of sound:

\[ M(v) = \frac{V}{c_0}. \quad (14) \]

### 2.2 The Control System Design Model

A linearized model of the missile dynamics was used in the control system design. This model neglects longitudinal dynamics, and assumes only rotational motion, since it is the key to control the missile normal acceleration (or pitch rate) in order to achieve the optimal flight path and eventually collision.

The linearized dynamics of the missile can be represented using the following equations [4, p. 76-78]:

\[ \begin{align*}
\dot{\alpha} &= q + \frac{Z_{\alpha}}{c_0} + \frac{Z_{\alpha \delta}}{c_0} \\
\dot{q} &= M_\alpha \cdot \alpha + M_\delta \cdot \delta \\
\dot{\delta} &= \frac{1}{\tau_\delta} (u_\delta - \delta)
\end{align*} \quad (15) \]

The elevator dynamics is now identical to the "real model". Also the second state equation of this model is quite similar to the more complex model, neglecting only the moment induced by the change in the missile’s mass. On the other hand, the nonlinearities of the drag and lift coefficients are simplified into constants $Z_\alpha$ and $Z_\delta$. Moreover, the missile velocity is assumed constant.

The pitch rate was chosen to be the output variable, mainly because the feedback loop using it was more easily constructed in the simulation model. Input of the model is the command signal to the elevator, $u_{\delta}$. 
<table>
<thead>
<tr>
<th>Description</th>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Atmosphere and earth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Air density</td>
<td>$\rho$</td>
<td>1.2249 kg/m$^3$</td>
</tr>
<tr>
<td>Acceleration due to gravity</td>
<td>$g$</td>
<td>9.80665 m/s$^2$</td>
</tr>
<tr>
<td>Speed of sound</td>
<td>$c_0$</td>
<td>340.3 m/s</td>
</tr>
<tr>
<td><strong>Missile dimensions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empty missile mass</td>
<td>$m_{\text{empty}}$</td>
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<tr>
<td>Initial mass of fuel</td>
<td>$m_0^f$</td>
<td>250 kg</td>
</tr>
<tr>
<td>Missile length</td>
<td>$l$</td>
<td>1.80 m</td>
</tr>
<tr>
<td>Missile surface area</td>
<td>$S$</td>
<td>0.150 m$^2$</td>
</tr>
<tr>
<td><strong>Missile rotational characteristics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Center of gravity of empty missile</td>
<td>$d_{\text{eq}}^\text{empty}$</td>
<td>0.38 l</td>
</tr>
<tr>
<td>Center of gravity of full fuel tank</td>
<td>$d_{\text{eq}}^f$</td>
<td>0.63 l</td>
</tr>
<tr>
<td>Missile center of pressure</td>
<td>$d_p$</td>
<td>0.625 l</td>
</tr>
<tr>
<td>Elevator surface center of pressure</td>
<td>$d_\delta$</td>
<td>0.92 l</td>
</tr>
<tr>
<td>Angle-of-attack moment coefficient</td>
<td>$C_{\alpha}$</td>
<td>-0.01</td>
</tr>
<tr>
<td>Elevator moment coefficient</td>
<td>$C_\delta$</td>
<td>-0.5</td>
</tr>
<tr>
<td><strong>Missile lift and drag</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lift coefficient</td>
<td>$C_L(\alpha)$</td>
<td>$2.93 + 0.34008M$ +0.2615M$^2$ + 0.0108M$^3$</td>
</tr>
<tr>
<td>Zero drag coefficient</td>
<td>$C_D_0$</td>
<td>0.45 – (0.04/3)M</td>
</tr>
<tr>
<td>Induced lift coefficient</td>
<td>$K$</td>
<td>0.053</td>
</tr>
<tr>
<td><strong>Missile control and fuel consumption</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Minimum throttle</td>
<td>$T_{\text{min}}$</td>
<td>0 N</td>
</tr>
<tr>
<td>Maximum throttle</td>
<td>$T_{\text{max}}$</td>
<td>100630 N</td>
</tr>
<tr>
<td>Fuel energy coefficient</td>
<td>$K_P$</td>
<td>-0.0003888 Ns/m</td>
</tr>
<tr>
<td>Thrust engine delay constant</td>
<td>$\tau_T$</td>
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</tr>
<tr>
<td>Maximum elevation angle</td>
<td>$\delta_{\text{max}}$</td>
<td>60$^\circ$</td>
</tr>
<tr>
<td>Elevator delay constant</td>
<td>$\tau_\delta$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 1: Numerical values for the "real model".

Throttle is not included in this model as an input. Constant velocity does not imply constant throttle, instead, one could define another control system, the purpose of which would be to keep the velocity constant. In this study we use a more simple approach, where throttle is dependent on the measured distance between the missile and the target, resulting, naturally, in a time-varying velocity (see chapter 4).

The linear state-space representation of the model can be derived from equation 15 as
\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{q} \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
\frac{2}{\theta} & 1 & \frac{2}{\theta} \\
M_{\alpha} & 0 & M_{\delta} \\
0 & 0 & \frac{1}{J}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
q \\
\delta
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{\tau_e}
\end{bmatrix} u,
\] 
\quad \text{(16)}
\]

where, by replacing numerical values, we obtain

\[
A = \begin{bmatrix}
-111.7 & 1 & -37.12 \\
-0.2393 & 0 & -41.96 \\
0 & 0 & 20.00
\end{bmatrix}
\quad \text{(17)}
\]

\[
B = \begin{bmatrix}
0 & 0 & 20.00
\end{bmatrix}^T
\quad \text{(18)}
\]

\[
C = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\quad \text{(19)}
\]

\[
D = 0
\quad \text{(20)}
\]

The transfer function of the system now becomes

\[
G(s) = \frac{-839.2 s - 9.358 \cdot 10^4}{s^3 + 91.72 s^2 - 2234 s - 4.787}.
\quad \text{(21)}
\]

### 2.3 The Guidance Law Model

The third missile dynamics model used in the study formed the basis for the guidance law being used. The purpose of this model is to describe the relationship between the missile and the target, making it possible to determine the optimal acceleration of the missile in order to achieve collision.

The guidance law used in this study was proportional navigation. This law can basically be derived from the following state-space model of the missile-target scheme [4, p. 367-368]:

\[
\begin{align*}
\dot{\lambda} &= \frac{1}{\sqrt{T^2}} z \\
\dot{z} &= T a_N
\end{align*}
\quad \text{(22)}
\]

Here \( \lambda \) is the line-of-sight to the target and \( z \) is the projected miss distance, if \( \lambda \) remains unchangeable. \( V \) is the missile velocity relative to the target, and \( T = T - t \) is "time-to-go", a quantity assumed to be known, and describing the time to collision. The model input is \( a_N \), the missile normal acceleration.

Both states are assumed to be measurable, and the target velocity is assumed constant. In spite of several of simplifications of this model, the guidance law derived from it has been proven to be very effective. Another upside is also the ease of its implementation. [4, p. 367-369]
3 Control System Design

3.1 General

Control design in this study was made using robust control techniques. Robustness of the resulting controller was essential because of the numerous uncertainties related to the task. First, the missile dynamics model was linearized for the control system design, simultaneously making many simplifications about the quantity and constantness of certain state variables. Second, the flying missile is exposed to different external disturbances in form of sensor noise and wind. Even the target behaviour can be considered as external noise, since its acceleration was not part of our model used in guidance law design. Last, the use of pitch rate as feedback variable instead of normal acceleration can be considered as an uncertainty-increasing factor.

Especially important from the point of view of this study were the internal disturbances resulting from simplified modeling, since they are known to exist and have a relatively large magnitude.

3.2 $H_\infty$ performance and (sub)optimal controllers

Robust performance of a controller is always a trade-off between the two conflicting requirements: robustness and performance. In the control design problem of this study, the design objectives are:

- Minimize the pitch rate error with respect to the reference signal
- Minimize the effect of disturbance to pitch rate
- Keep the control signal mostly inside certain boundaries

It is obvious that first two objectives are conflicting. In addition to them, we have a state constraint that again downgrades the controller performance, limiting the maximum controller gain.

These requirements can be dealt effectively using the $H_\infty$ framework. The principle of $H_\infty$ optimal design is to build a controller, which minimizes some signals in the closed-loop system. The quantities of these signals can be represented in terms of norms, thus transforming the $H_\infty$ control design into a norm-minimizing task. [5]

The $\infty$-norm of a signal can be represented as the least upper bound of its absolute value: [5, p. 12]

$$\|u\|_\infty := \sup_t |u(t)| .$$

(23)

The original plant block diagram has to be augmented to include the important signals (the norms of which are to be minimized). This is done by including
disturbance and noise as inputs, and adding specific weighing transfer functions into the model. The role of the weighing functions is both to normalize the requirements and give certain frequencies specific importance; for example, we want the system output to track the reference signal only in low frequencies, whereas high frequency inputs are most probably noise, and can be neglected. [6, p. 85-89]

The modified open-loop system for the $H_\infty$ minimizing task is presented in figure 4. $P$ represents our plant, and there are three weighting functions included: $W_\varepsilon$ weights the performance of the controller, $W_u$ weights the control signal quantity, and $W_d$ weights the external disturbance. It is worth pointing out that the two first weights differ from last one: $W_\varepsilon$ and $W_u$ weigh the important frequencies of two system outputs, and hence shaping the output signals, the norms of which are involved in the $H_\infty$ minimizing task. $W_d$, on the other hand, describes the frequency content of the external disturbance.

\[
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix}
\] . \quad (24)

Figure 4: Open-loop system for $H_\infty$ optimizing task.

The feedback loop is constructed from $y$ to $u$, whereas signals $\hat{u}$ and $e$ are the ones to be minimized by the resulting controller.

This extended MIMO system can be described as a partitioned system matrix of the form [6]
Assuming the following:

i) \((A, B_1)\) is controllable and \((C_1, A)\) is observable;

ii) \((A, B_2)\) is stabilizable and \((C_2, A)\) is detectable;

iii) \(D_{12} \begin{bmatrix} C & D_{12} \end{bmatrix} = [0I];\)

iv) \( \begin{bmatrix} B_1 & D_{21} \end{bmatrix} D_{21}^* = [0I].\)

It has been shown [6, p. 270-277] that there exists an admissible controller such that the robust performance norm associated with the open-loop system reaches a level below \(\gamma\), that is, \(\|T_{zw}\|_\infty < \gamma\), if and only if the following conditions hold:

i) \(H_\infty \in \text{dom}(\text{Ric})\) and \(X_\infty := \text{Ric}(H_\infty) > 0;\)

ii) \(J_\infty \in \text{dom}(\text{Ric})\) and \(Y_\infty := \text{Ric}(J_\infty) > 0;\)

iii) \(\rho(X_\infty Y_\infty) < \gamma^2,\)

where

\[
H_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^* - B_2B_2^* \\ -C_1^*C_1 & -A^* \end{bmatrix} \quad \text{(25)}
\]

\[
J_\infty := \begin{bmatrix} A^* & \gamma^{-2}C_1^*C_1 - C_2^*C_2 \\ -B_1B_1^* & -A \end{bmatrix}. \quad \text{(26)}
\]

These conditions holding, one such controller is [6, p. 271]

\[
K_{sub}(s) := \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix}, \quad \text{(27)}
\]

where

\[
\hat{A}_\infty := A + \gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty + Z_\infty L_\infty C_2, \quad \text{(28)}
\]

\[
F_\infty := -B_2^*X_\infty, \quad \text{(29)}
\]

\[
L_\infty := -Y_\infty C_2^*, \quad \text{(30)}
\]

\[
Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}. \quad \text{(31)}
\]

Since it is virtually useless to compute all the \(H_\infty\) optimal controllers [6, p. 269-270], we shall concentrate on this realization of a suboptimal controller (27). The system matrix \(G(s)\) defined, this controller can easily be solved with Matlab’s \texttt{hinfsyn} command. Before entering this stage, we first have to take a closer look at the weighing functions.
3.3 Weighing Function Selection

Equation (27) is guaranteed to result in an $H_{\infty}$ suboptimal controller, given the weights $W_u$, $W_d$ and $W_e$. The question arising is common to all weighted optimization problems: How to choose the performance weights to achieve the "best among the optimal solutions"?

The weighing function selection is generally considered as a rather demanding task [5, 6]. The weight selection becomes, thus, a massive iteration process, where the performance of the resulting controller is evaluated after each weight adjustment. The first task considering the weights now becomes choosing the correct shape of the frequency response magnitude plot.

$W_e$ is the performance weight, and should be large in those frequencies, where a good performance is desired. Usually this frequency band is the low frequencies, as also in this case. This way the resulting controller attempts to track low frequency reference signals, but high frequency signals, which are most likely noise, are neglected. $W_e$ has then the same shape than a low-pass filter, the magnitude rolling off at certain frequency.

$W_u$ describes the frequencies where the control signal has to be kept small. Since high-frequency controls are those, which the controller is assumed to neglect to certain extent, there is no reason to downgrade the performance of the controller by restricting the controls also in these frequencies. The control weight $W_u$ has accordingly the same shape than $W_e$: controls are restricted, but only at low frequencies.

Transfer functions for $W_e$ and $W_u$ now have the following generalized form:

$$W_{e,u}(s) = \frac{\mu}{1 + \kappa s} \quad (\mu, \kappa \in \mathbb{R}) \ .$$

(32)

The disturbance weight $W_d$ is used to describe the frequency content of the disturbance. Usually this can be assumed to be high at high frequencies, and roll off at low frequencies [5]. The weight $W_d$ thus becomes a high-pass filter. Since the algorithm used for the controller synthesis requires the weighing functions to be strictly proper, we add a far-away pole, achieving

$$W_d(s) = \frac{\mu s + 1}{\epsilon s + \kappa} \quad (\mu, \kappa \in \mathbb{R}) \ ,$$

(33)

where $\epsilon \ll \max(\mu, \kappa)$.

After determining the shape of the magnitude plots of the weights, the parameters in their transfer functions were chosen after careful iteration. The objective of the iteration was to gain a pitch rate of 1 rad/s in a time of less than 1 s, with a reasonable overshoot. Therefore, the performance of the controller was evaluated based on the response to a step input of 1 rad/s.

After each weight adjustment the $H_{\infty}$ optimal controller was solved, and its performance evaluated using step response and state trajectory plots. The best
achieved weights resulted in a sufficient compromise between the performance of the controller, in other words, the characteristics of the step response, and the magnitude of the elevation angle. The step response is shown in figure 5, and the corresponding control signal and the elevation angle trajectory in figure 6.

Figure 5: The closed-loop step response with the chosen controller.

Figure 6: Elevation angle trajectory in step response with the chosen controller.

The weights resulting for the best performance were
\begin{align*}
W_c &= \frac{0.006s + 90}{15s + 0.1} \quad (34) \\
W_u &= \frac{0.72}{0.45s + 1} \quad (35) \\
W_d &= \frac{0.15s + 15}{0.0001s + 10} \quad (36)
\end{align*}

The frequency response magnitude curves of these weights are shown in figure 7.

Figure 7: Frequency response magnitude plots for the performance weights.

### 3.4 $H_\infty$ optimal solution

Considering the determined weights, (33-36) the transfer matrix of the weighed open-loop system becomes

\[
G(s) = \begin{bmatrix}
-110.0 & 1.000 & -37.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.240 & 0 & -42.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 20.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.000 & 0 & -0.0067 & 0 & -18290 & 0 & 0 & 2.000 \\
0 & 0 & 0 & 0 & -2.200 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -100000 & 16000 & 0 & 0 \\
0 & 0.00040 & 0 & 3.000 & 0 & -3.700 & 0.600 & 0 & -0.0004 \\
0 & 0 & 0 & 0 & 1.600 & 0 & 0 & 0 & 0 \\
0 & 1.000 & 0 & 0 & 0 & -9150 & 1500 & 0 & 0
\end{bmatrix}
\quad (37)
\]
The $H_\infty$ controller was solved with Matlab’s \texttt{hinfsyn} command, using tolerance 0.01. The resulting controller was

$$C(s) = \frac{P(s)}{Q(s)}, \quad (38)$$

where

$$P(s) = s^6 + 3.327 \cdot 10^4 s^5 + 1.051 \cdot 10^9 s^4 + 1.238 \cdot 10^{11} s^3 + 7.689 \cdot 10^{11} s^2 + 1.11 \cdot 10^{12} s + 2.375 \cdot 10^9 \quad (39)$$

$$Q(s) = s^6 + 2.459 \cdot 10^4 s^5 + 1.344 \cdot 10^8 s^4 + 3.491 \cdot 10^{10} s^3 + 2.299 \cdot 10^{12} s^2 + 5.005 \cdot 10^{12} s + 2.828 \cdot 10^{10} \quad (40)$$

### 3.5 Controller Performance Analyzed

While choosing the weighing functions, the performance of this controller was examined using intuitive step response and elevation angle trajectory plots. The performance can more formally be studied by such concepts as nominal performance, robust stability and robust performance.

The formal treatment of these concepts requires us to define an uncertainty model for the plant transfer function. Without considering, how these uncertainties are generated, we can refer to the set of all possible plant transfer functions with \( \Pi \), discussed in more detail below.

The relationships of the three performance measures are shown in figure 8. Nominal performance means the controller’s performance with respect to the nominal plant, whereas robust stability indicates closed-loop system stability for all plants \( \Pi \). Last, robust performance measures the controller’s performance with respect to the whole model set \( \Pi \). Clearly, robust performance is the strictest measure. \[6, 5\]

![Diagram showing the relationships of the three performance measures](image)

Figure 8: Mutual relationships of the three performance measures used.

Furthermore, we need two transfer functions describing the closed-loop system:
\[ S(s) = \frac{1}{1 + P(s)C(s)} \]  

(41)

is the sensitivity function, and

\[ T(s) = 1 - S(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} \]  

(42)

is the complementary sensitivity function. Now these functions defined, we can study the robust performance of the controller with respect to some uncertainty model. Because of the complexity of our model (see equation 15), the uncertainty is best to be modeled as unstructured. This means that the frequency contents of the uncertainty are enclosed inside a curve, which then represents the "worst case uncertainty".

For simplicity, let us assume the uncertainty to be of multiplicative type. This is a safe assumption, since the performance will eventually be studied also using simulations, and the results of this analysis are only suggestive. For a SISO system, multiplicative perturbation can be described as

\[ \Pi := (I + W_2 \Delta) P , \]  

(43)

where \( W_2 \) is a transfer function describing the frequency contents of the perturbation, and \( \Delta \) is the normalized perturbation satisfying \( \| \Delta \| \leq 1 \).

The controller performance in the sense of nominal performance, robust stability and robust performance can be measured in terms of weighted norms. For a SISO system and multiplicative perturbation model, these norms are as follows:

- **Nominal performance**: \( \| W_1 S \|_\infty \)  
- **Robust stability**: \( \| W_2 T \|_\infty \)  
- **Robust performance**: \( \| W_1 S \| + \| W_2 T \|_\infty \)  

Performance is guaranteed, if the corresponding norm has a value less than 1. [6, p. 147-149]

It is most convenient to have the already defined performance weight to weigh the nominal performance norm, that is, \( W_1 = W_e \). Since our controller design was made so that performance corresponding this criterion was guaranteed, it is no surprise that the nominal performance norm satisfies the performance criterion:

\[ \| W_e S \|_\infty = 0.65 . \]  

(47)
For robust stability and robust performance, the analysis becomes again a pursuit after admissible weighing functions, with which the norms satisfy the performance criteria. For robust stability, the following weighing function was found:

\[ W_{2}^{RS} = \frac{0.1s + 1.5}{6} \],

(48)

for which the performance norm has the value

\[ \|W_{2}^{RST}\|_{\infty} = 0.96 \].

(49)

Singular value plots for transfer functions \( W_{e}S \) and \( W_{2}^{RST} \) are shown in figure 9. There we see that for both transfer functions, the peak occurs at quite similar frequencies. This implies that this frequency range could be important in enhancing the controller’s performance.

![Singular values](image)

**Figure 9:** Singular value plots for nominal performance and robust stability transfer functions.

Nominal performance and robust stability are important features of the designed controller, but even together they don’t imply robust performance, that is, performance for the whole plant set II. Indeed, for robust performance, an admissible weighing function was not found in spite of extensive iterations. Even for a weighing function

\[ W_{2}^{RP} = \frac{0.1s + 1}{500} \]

(50)
the robust performance norm has a value of $\| \cdot \|_\infty = 3.06$. The failure of the controller with regard to this criterion can be seen in the figure 10, where the frequency response magnitudes of the perturbation weighing functions are illustrated. We see that for example at frequency $w = 10\,\text{rad/s}$ even as small perturbation as of magnitude $|W_2\Delta| = 0.01$ results in bad performance. On the other hand, robust stability is guaranteed for relatively large perturbation magnitudes.

![Frequency response](image)

Figure 10: Frequency response magnitude plots for perturbation, for which robust stability is achieved. In addition, the magnitude is plotted for the perturbation, for which a robust performance level of 3.06 is achieved.

### 3.6 Model reduction

#### 3.6.1 General

Since the resulting controller is of order 6, it is very probably beneficial to try reducing its order. Lower-order controller is generally easier - and that way cheaper - to implement than the full-order optimal controller.

The algorithm used in the model reduction is practically divided in two parts. First, the balanced realization is calculated, and based on the controllability and observability matrices of this realization, the model order is further reduced.

#### 3.6.2 Balanced Realization

The balanced realization for a general, linear state-space model
\[
\begin{aligned}
\begin{cases}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{cases}
\end{aligned}
\] (51)

is formulated using a state variable transformation of the form

\[
\tilde{x} = Tx .
\] (52)

The algorithm used results in a transformed state variable \(\tilde{x}\) so that the balanced system has the observability and controllability matrices being diagonal and identical:

\[
W_o = W_c = \text{diag}(g) .
\] (53)

The diagonal items \(g_i\) can be used to delete nonsignificant states from the model. The algorithm for finding the state transformation is described in [7] and not treated here.

The diagonals for the balanced realization of the \(H_\infty\) optimal controller (38) were calculated using Matlab, and are shown below:

\[
g = \begin{bmatrix}
4.0209 & 3.8902 & 0.5186 & 0.0689 & 0.0007 & 0.0004 \\
\end{bmatrix} .
\] (54)

Clearly, the two last states can be deleted, since the corresponding diagonals are significantly smaller compared to the others. Being more confident, even the next two states could be deleted, resulting in a controller of only second order. This would definitely be a good goal, so we shall build both the two reductions and study their characteristics in the frequency plane.

### 3.6.3 Reduced Model Formulation

The method used for model reduction produces a reduced-order model with a steady-state (step) response matching to the original model. The principle of the method is to make the derivatives of the states to be deleted zero, and then to solve the remaining states [7]. The model is first partitioned into state groups \(x_1\) and \(x_2\), the former being preserved and the latter being deleted:

\[
\begin{aligned}
\begin{cases}
\dot{x}_1 &= A_{11} x_1 + A_{12} x_2 \\
\dot{x}_2 &= A_{21} x_1 + A_{22} x_2 \\
y &= C_1 x_1 + C_2 x_2 + B_1 u + B_2 u
\end{cases}
\end{aligned}
\] (55)

After setting \(\dot{x}_2\) to zero, the reduced-order model can be represented as

\[
\begin{aligned}
\begin{cases}
\dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\
y &= [C_1 - C_2 A_{22}^{-1} A_{21}] x + [D - C_2 A_{22}^{-1} B_2] u
\end{cases}
\end{aligned}
\] (56)
Using this method, the resulting reduced-order controllers for (38) are

\[ G_4(s) = \frac{0.9978s^4 + 3.314 \cdot 10^4 s^3 + 1.039 \cdot 10^9 s^2 + 4.382 \cdot 10^9 s + 9.384 \cdot 10^6}{s^4 + 2.431 \cdot 10^4 s^3 + 1.307 \cdot 10^8 s^2 + 1.973 \cdot 10^{10} s + 1.117 \cdot 10^8} \]  
\[ (57) \]

for the fourth-order controller, and

\[ G_2(s) = \frac{-0.1773s^2 + 5.315 \cdot 10^4 s + 8.043 \cdot 10^4}{s^2 + 6758s + 9.577 \cdot 10^5} \]  
\[ (58) \]

for the second-order controller, respectively.

### 3.6.4 Model reduction comparison

The bode diagrams of the original and reduced-order controllers are shown in figure 11. In the figure we can see that the frequency response of the fourth-order model is virtually identical to the original, full-order, controller. This implies that at least the two last states can be deleted.

![Bode Diagram](image)

Figure 11: Frequency responses for the original and reduced-order controllers.

However, also the greater reduction results in a very similar frequency response. The difference to the original controller model appears only in very high frequencies, which again implies that even model order reduction of this level could be acceptable. Since the controller is specifically not meant to track high frequency inputs, we decide to choose the second-order controller.
Figure 12: Nyquist plots for the original (dotted line) and second-order (solid line) controllers.

As seen in the nyquist plots in figure 12, the reduced model has in fact better gain margin than the full-order model. This refers to better robustness, although most possibly with the cost of lost performance.

4 The Guidance System

4.1 Pitch rate command

The missile-target model that was the basis of the guidance law derivation was presented in chapter 2.3. In principle, several different feedback guidance laws can be derived from this model using dynamic optimization.

Whereas in the autopilot design a method was used to minimize an $H_\infty$ norm of the system, the guidance law derivation involves minimizing an $H_2$ norm. A special case of proportional navigation is achieved, if the optimization criterion is chosen to be

$$J = k^2 z^2(T) + \int_0^T a_N^2(\tau) d\tau,$$

where $k$ describes the importance of the miss distance at final time $T$. The derivation of the proportional navigation guidance law includes setting $k \to \infty$, which practically means that the target has to be hit at all costs. Solving the optimal guidance law with this addition results in
\[
a_N(t) = -3V(t)\dot{\lambda}(t),
\]

(60)

where \(\dot{\lambda}\) is the line-of-sight angle and \(V\) the missile velocity relative to the target.

Replacing the constant with \(N\) gives us the general proportional navigation formula. Usually a value between \(3 < N < 5\) used. [4, p. 367-369]

The guidance law used with the control system derived in chapter 3 is a modified version of this general proportional navigation law: the normal acceleration is replaced with the pitch rate. Intuitively, this corresponds to using normal acceleration, but it requires us to find a suitable value for the static gain constant. After some simulations, a decent value was found to be \(N = 0.015\).

4.2 Throttle command

The proportional navigation now derived considers only the pitch rate command delivered to the autopilot. In addition, the desired throttle force should be determined. Not much concern was laid on this, but instead a simple approach was taken.

A good starting point was to make the throttle directly proportional to the distance between the missile and the target. The throttle should not, however, go to zero when the target is "near", so the relation was made logarithmic. Again after some simulations, the following throttle feedback law was found to be good enough:

\[
T(d(t)) = T_{max}\frac{\log(d_0)}{\log(d(t))},
\]

(61)

where \(d_0\) is the initial distance between the target and the missile. A plot of the throttle command is shown in figure 13.

5 Control System Performance Study by Simulations

5.1 The Simulation Framework

The "real model" was built using Simulink, and it was equipped with the designed controller and the guidance system in order to simulate the missile performance in a target encounter situation. A block diagram of the simulation model is shown in figure 14.

The missile body axis model has airspeed, angle-of-attack, elevator angle and throttle force as its inputs. Outputs, missile velocity and rotational motion, are then transformed into earth reference axis velocity and orientation. The missile
Figure 13: The throttle command as a function of distance between the target and the missile.

actuators are described in a separate block getting its inputs from the guidance system. In this study, the observation system is a dummy block, observing the missile and target states with a perfect accuracy.

5.2 Simulations performed

5.2.1 General about Simulation Conditions

In all the simulations, the missile was launched at the origin of the $xz$-plane. The target initial height was 500 m in all the simulations. In the scenarios where the target was moving away, its initial $x$ coordinate was 2000 m. In approaching scenarios, the target started at 5000 m, unless otherwise mentioned.

Initial velocity of the missile was 10 m/s, and of the target 200 m/s. The target velocity was assumed constant.

5.2.2 Effect of Model Order Reduction

First the effect of controller order reduction to the performance of the missile was studied. The target was assumed to approach the missile launching point making sinusoidal movements. Missile and target trajectories using both the full-order and reduced-order controllers are shown in figure 15.
Figure 14: The simulation model for the missile-target encounter situation.

We see that both controllers result in collision. The original full-order controller is slightly more effective in steering the motion, resulting in the missile turning towards the target more than its counterpart.

The error of pitch rate realization compared to the reference signal are shown in figure 16 for both the reduced and full-order controller. There we see that neither one of the controllers tracks the signal perfectly, but full-order controller reaches a significantly better performance.

These simulation results are in line with the analysis presented in chapter 3.6.4: the reduced-order controller has better robustness than its counterpart, but loses in performance.

Based on these simulations, we can conclude that the reduced-order controller is enough for our purposes. It appears, in fact, that in the simulations performed,
Figure 15: Flight paths of missile with the original (solid) and reduced-order (dashed) controllers.

Figure 16: Pitch rate error of the missile with the original and reduced-order controllers.

none of the non-collision situations could have been saved using the full-order controller, but the missile failure was due to other reasons.
5.2.3 Effect of Target Maneuvering

The missile performance was evaluated using simulations with respect to different kinds of target maneuvering. Flight paths resulting from different kinds of maneuver patterns are shown in figures 17-19. The target was assumed to be moving away from the launching point of the missile.

![Flight path diagram](image_url)

Figure 17: Flight paths; constant target velocity.

In the situations illustrated in these figures the missile performance is rather good: collision is achieved. Pitch rate compared to the reference signal produced by the guidance system is illustrated for third of the simulation setups in figure 20, and elevation angle trajectory in figure 21.

From the pitch rate plot we see that the tracking is actually far from being accurate. This seems to be enough, however, for reaching collision. Because of the high uncertainties and modeling errors, better tracking might easily result in instability of the missile rotational motion.

The elevation angle plot gives us a good measure for the effect of the control weight $W_a$. The elevation angle gets saturated only in the very end of the missile flight, a result of greater steering movements required when being very close to the target; this saturation is a characteristics of the guidance law used, and could be diminished with a different guidance law design.

However, increasing the frequency of the target sinusoidal maneuvers results in the failure of the missile. As we can see in figure 22, the minimum distance of the missile and the target is 155 meters, which implies a total failure of the control system. This suggests that our control system is not robust enough to damp high-frequency command signals effectively; this can be verified by examining
Figure 18: Flight paths; target performing sinusoidal maneuvers.

Figure 19: Flight paths; target performing sinusoidal maneuvers.

figure 23, which represents the reference signal and the pitch rate realization. The high-frequency signals result in missile instability.

To increase missile robustness in order to achieve better performance in this setup, the control design could be conducted again, having the performance weight $W_c$ bode magnitude plot adjusted more to the left, that is, having a
smaller roll-off frequency. The effect of this would be that the control system performance around the original roll-off frequency becomes less important, and thus the missile would not react so delicately in inputs with this frequency.

Studies were also made with an approaching target. Simulations shown in figures 24 and 25 illustrate, how even a non-maneuvering target can be missed, if
Figure 22: Flight paths for a simulation, where collision is avoided.

Figure 23: Pitch rate and reference signal for a simulation, where collision is avoided.

the launching point is too close to the target considering its velocity.

This is clearly a result of greater derivative of change in the line-of-sight angle, compared to the the situation where the target is moving away. Based on the guidance law (60), this results in greater reference signals to the control
system, and again saturation of the controller surface. Figure 26 illustrates the pitch angle trajectory and reference corresponding to the second simulation, and elevation angle plot in figure 27 show the saturation.

Figure 24: Flight paths; approaching non-maneuvering target.

Figure 25: Flight paths; approaching non-maneuvering target.
Figure 26: Pitch angle and reference signal for a simulation.

Figure 27: Elevation angle trajectory for a simulation.
One more simulation with an approaching target is shown in figure 28. The target performs sinusoidal movements corresponding to the case in figure 19. This case leads to collision, which implies that missile nominal performance is not affected by the target orientation itself.

![Flight paths](image)

**Figure 28:** Flight paths; target performing sinusoidal movements.
<table>
<thead>
<tr>
<th>Target velocity (m/s)</th>
<th>Loop radius (m)</th>
<th>Target $a_N$ (m/s²)</th>
<th>Minimum distance (m)</th>
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</thead>
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</tr>
<tr>
<td>150</td>
<td>30</td>
<td>750</td>
<td>13.53</td>
</tr>
</tbody>
</table>

Table 2: Simulation results with a dodging target.

Last, simulations were made in order to find the boundaries of missile performance in terms of G forces the target has to produce to be able to avoid the missile. In these simulations, the target was moving away, starting at 2000 meters. When the missile was approximately 300-400 meters away, the target performed a single loop dodging maneuver lasting 0.2-1.0 seconds, depending on the target velocity.

Simulation results and performed target normal accelerations are shown in table 2, as well as the radius of the target loop. A close-up of the minimum-distance point of one simulation that ended up in target miss is shown in figure 29.

![Flight paths](image)

Figure 29: A detail of flight paths; target performing a dodging maneuver with a normal acceleration of 330 m/s².

These results show that our missile is very capable of hitting a fighter airplane.
The normal accelerations needed to avoid the missile exceed greatly the tolerance of best fighter pilots [8], indicating, that a fighter can’t avoid a hunting missile by performing simple dodging maneuvers. Especially faster-moving targets find the necessary normal accelerations very high. Of the simulated cases the only one where the pilot might have a chance to survive the maneuver is when the fighter moves at 60 m/s. There the normal acceleration is 90 m/s², or 9.2 G.

5.2.4 Reasons behind Failure Analyzed

It is worth taking a deeper look at the reasons behind the missile failure in the simulations performed. Three main causes of miss were found, two of which result from the control system’s performance.

First of all, it is easy to see that the initial conditions of the missile have a great influence on the performance. Figure 25 shows that even a non-maneuvering target can be missed, if the closing velocity is too high. This is the case usually when the target is too close to the missile launching point and approaching.

Further studies would be needed to determine, what distance exactly is "too close". Always trying to hit an away-moving target might not be a clever strategy either, since the missile has a limited amount of fuel, and the flight can be sustained only for a limited amount of time. From table 1 and equation (13) we can calculate, that if the missile uses approximately 80% of its throttle capacity, the thrust can be sustained only for

\[ t = \frac{m_{fluid}}{0.80 \cdot K_pT_{max}} = 8.2 \text{ seconds}. \] (62)

This viewpoint was neglected in this study, assuming the target velocity constant. A further study could be done to determine the optimal launching point and initial flight path angle of the missile with respect to maximizing the probability of collision.

The problems with the controller design deal with either the lack of robustness, or the lack of performance. The robustness problem appeared with a target maneuvering at a high frequency (see figure 22), the controller attempting to track high-frequency reference signals, and eventually resulting in instability. The same could also be tackled by a more sophisticated guidance law design.

Performance problems are due to too high reference signals: when the reference is small enough, even a poorer performance of the controller results in collision in the cascading overall feedback loop, consisting of the autopilot and the guidance system. But when the reference is too high (see figure 28), the tracking becomes poor, and the missile misses the target.

These limits of robustness and performance are essential considering the control design. Since both the robustness and performance lack some effectiveness, and the missile performance in the simulations is "rather good", we can quite safely conclude that a sufficient trade-off between these design goals is achieved. If we felt that the controller is biased towards either robustness or performance,
we could repeat the design process by adjusting the weights correspondingly. Based on the simulations made, this design reaches a good balance between the requirements for the control system.

6 Conclusions

$H_\infty$ framework was an advantageous approach for this control system design task, where the objectives were conflicting, and uncertainty of the plant was further increased by model reduction. A controller, which was designed on a significantly simpler model resulted in an excellent in-action performance in simulations with the more realistic model. We can therefore conclude that the simplified linear model of the missile dynamics was enough for controller design.

The simulations show, that a decent trade-off between robustness and performance was achieved. Robust performance was not met, however, which implies that the control design could be further improved by other methods, like $\mu$ synthesis.

Robust stability appears to be crucial among the performance criteria, because the guidance system compensates poor performance of the autopilot, but the loss of stability can't be corrected. Because of the similar overall feedback loop among all guided missiles, the importance of robust stability over robust performance is likely to be universal.

Further studies could be made to find out the controller performance under sensor noise and wind conditions. Because of the high variation in missile velocity compared to the constant value used in control design, the performance under windy conditions is likely to be good, however. The most important open question after this study is, if the missile model now used was physically realistic. In any case, the design procedure could be repeated for any missile data in order to find the limits of that physical missile's capability.

References


